Linking Numbers

Final Project of

Introduction to Geometric Topology

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POSTECH MATHEMATICS

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Abstract

During the spring semester, we studied about two different ways to view topological object. The first one is an algebraic approach. By the notion of fundamental group, we learned how to calculate the fundamental group of given object such as Seifert-van Kampen theorem. And those languages helped us to distinguish knots by Wirtinger presentation, and Alexander polynomial of a given knot.

On the other hand, the second way was an differentiable approach. By the notion of smooth map and tangent spaces, we learned about smooth manifolds and diffeomorphism between them. Regular values and critical values of a smooth map gave us some nice theorems like Sard-Brown theorem. And those techniques also helped us to discuss about orientation and degree of a smooth map.

In this project, we will study about one of the cases where the algebraic approach and differentiable approach coincides, which is called a linking number. At first, by an algebraic approach, we will define what a linking number is. And we will figure out some nice properties of a linking number. We will find how to calculate linking number from the link diagram, and also prove that the linking number is an invariant. By that, we can distinguish equivalent and non-equivalent links.

After that, we will define linking number from differentiable approach. By defining a smooth map, we will define a linking number as a degree of a map. And again, we find some nice properties of it. Later, we will prove that the algebraically defined linking number and differentiably defined linking number are actually the same.

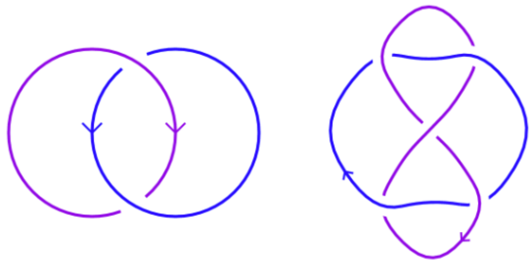
1 Introduction, background, and main results

Definition 1.1. A map $f : X \to Y$ is called **embedding** if $f : X \to f(X)$ is a homeomorphism

Definition 1.2. A **knot** in $\mathbb{R}^3$ is the image of an embedding $S^1 \to \mathbb{R}^3$. In other words, a knot is a simple closed curve in $\mathbb{R}^3$

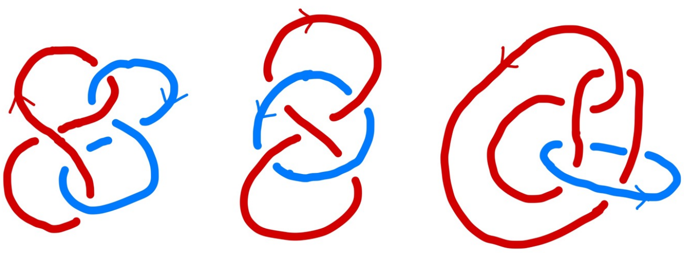
Definition 1.3. An m**-component link** in $\mathbb{R}^3$ is the union of m disjoint oriented knots in $\mathbb{R}^3$

Note that links are oriented. In this paper, we will study about only 2-component links, which is the most fundamental case. Figures in the below are some examples of a 2-component link.

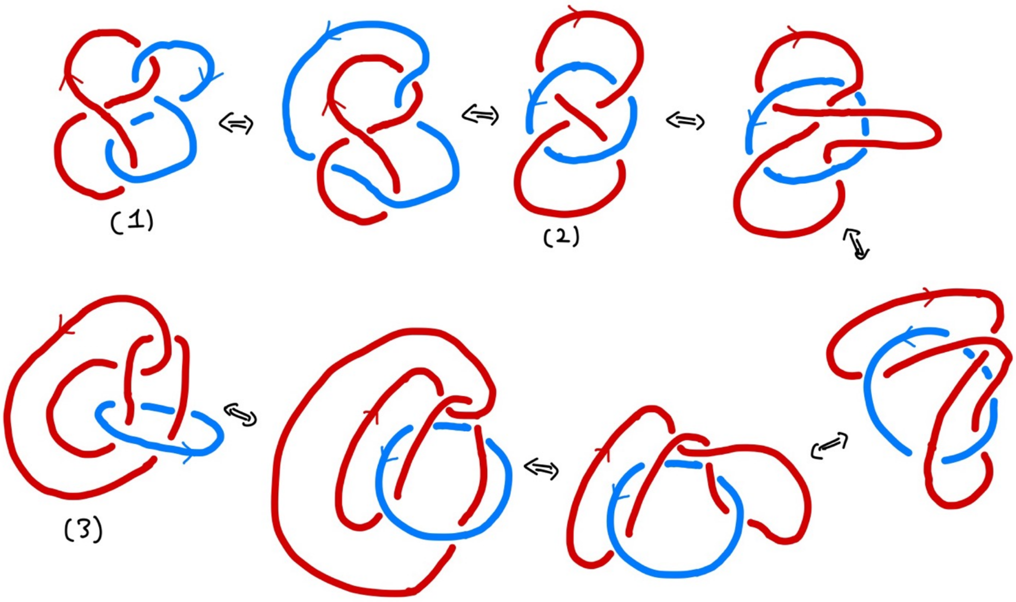


Definition 1.4. Two knots K and J in $\mathbb{R}^3$ are **equivalent** if there exists a homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(K) = (J)$.

By choosing how to draw a diagram of link, some equivalent links could be draw in different way, as the diagram of knots did. Those variations confuses us to distinguish whether the two links are equivalent or not. For example, three link diagrams in the below seems to be different.



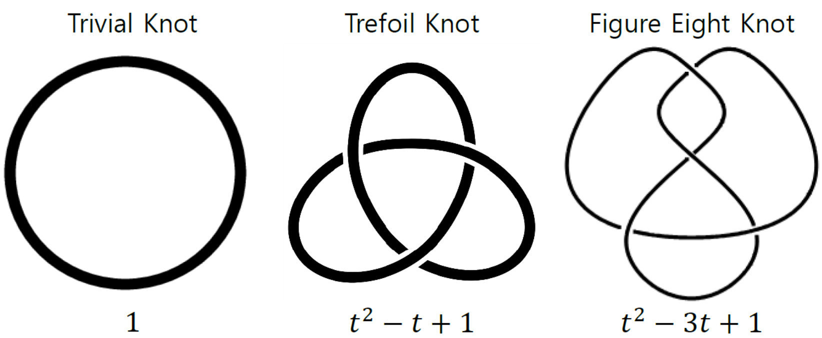
However, they are equivalent.



Drawing could be one of the way to distinguish equivalent knots and links, but there are some problems. One problem is that finding such transforming diagram is not always easy, and the other problem is that proving two links are not equivalent is much more harder.

But also directly proving from the mathematical definition whether the two knots or links are equivalent or not is hard. Finding such homeomorphism directly, or proving that there is no such homeomorphism, is both a challenging problem.

Instead, we learned some invariants which could help distinguishing knots. From the knot diagram, we could express the fundamental group of complement of knots, which is called 'Wirtinger presentation'. From that, we could calculate the Alexander polynomial of a knot. If two Alexander polynomial of given knot diagrams are different, we could conclude that two knots are not equivalent. For example, some non-equivalent knots and their Alexander polynomials are in below.



Note that the inverse is not true : some non-equivalent knots could have same Alexander polynomial. This tells us that distinguishing the equivalent and non-equivalent knot is still a challenging problem. However, Alexander polynomial is a powerful technique to distinguish non-equivalent knots. There are also other polynomial, called 'Jone's polynomial', which was not covered in our class.

So, like the Alexander polynomial for the knot, we will find an invariant for the 2-component link to distinguish whether the two 2-component links are equivalent or not. Those are called linking numbers. At first, by an algebraic approach, we will define what a linking number is, and find out some nice properties of a linking number. We will learn how to calculate linking number from the link diagram, and also prove that the linking number is an invariant. After that, we will define linking number from differentiable approach, which is a degree of some smooth map. In the end, we will prove that the algebraically defined linking number and differentiably defined linking number are actually the same.

2 Algebraic definition of the linking number

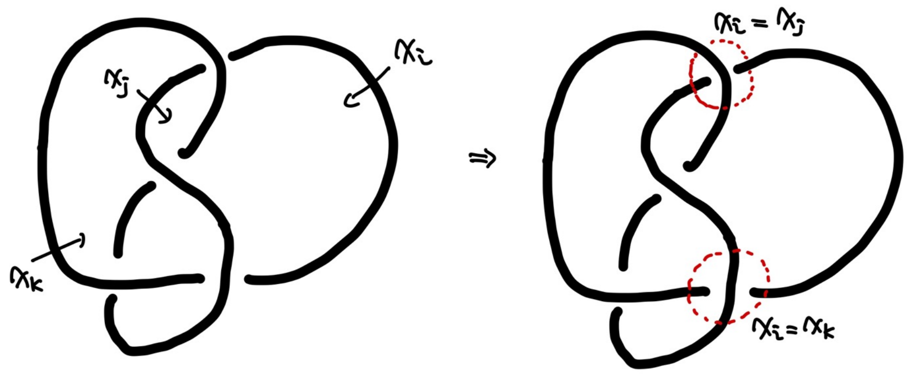
Definition 2.1. Suppose that K is represented by a knot diagram with n crossings. A **Wirtinger presentation** of a knot complement is a presentation of a fundamental group of knot complement where the n generators match with each strands, and n relators match with each crossing. i.e,

$$\pi_1(\mathbb{R}^3 - K) =< x_1, x_2, \cdots x_n \mid x_{a_1}x_{c_1}x_{b_1}^{-1}x_{c_1}^{-1}, \ \cdots , \ x_{a_n}x_{c_n}x_{b_n}^{-1}x_{c_n}^{-1} >$$

For every knot diagram, its fundamental group of knot complement can be expressed by Wirtinger presentation. Note that one of the relators could be deleted, as we learned in the class.

Theorem 2.2. For any knot $K \in \mathbb{R}^3$, $\pi_1(\mathbb{R}^3 - K)_{ab} \cong \mathbb{Z}$.

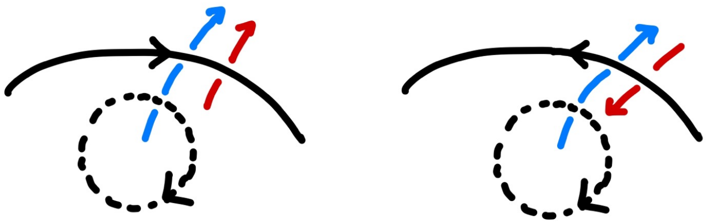
Proof. Let n be a number of strand of diagram of K. Denote the Wirtinger presentation of given diagram by $\pi_1(\mathbb{R}^3 - K) =< x_1, x_2, \cdots x_n \mid x_{a_1}x_{c_1}x_{b_1}^{-1}x_{c_1}^{-1}, \ \cdots , \ x_{a_n}x_{c_n}x_{b_n}^{-1}x_{c_n}^{-1} >$. After the abelianization, each relator $x_{a_i}x_{c_i}x_{b_i}^{-1}x_{c_i}^{-1}$ becomes $x_{a_i}x_{b_i}^{-1}$, which means $x_{a_i} = x_{b_i}$. Since every strand of a knot diagram has two endpoints, the generator which corresponds to the strand is contained in at least two relators, which corresponds to two endpoints.



So, after abelianization, all generators become equal, and no relators are left. Therefore, the abelianization of $\pi_1(\mathbb{R}^3 - K)$ becomes $< x \mid \cdot >$, which equivalent to the infinite cyclic group.

□

Note that knots could have two kinds of orientations. After the abelianization, a generator t of $\pi_1(\mathbb{R}^3 - K)_{ab} \cong \mathbb{Z}$ is canonically determined from the orientation of a knot. It is enough to consider the orientation of a single strand in knot diagram, since the orientation of whole knot can be determined by the orientation of a strand. Moreover, since every generators are same after abelianization, it is enough to look at just one strand to determine t.

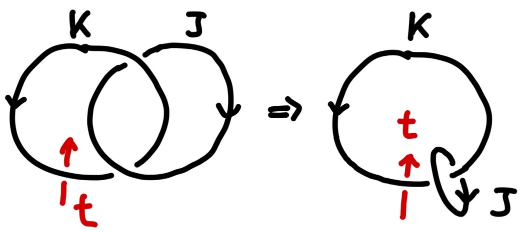


Using the right hand rule, we can choose the generator as the blue arrow in the above figure. If the orientation of the strand, the red arrow, coincide with the generator like the left figure case, the generator is same with the original one. If the orientation is reversed like the right figure case, then we get the generator as the inverse of the original generator. By those way, we can choose the generator from the orientation.

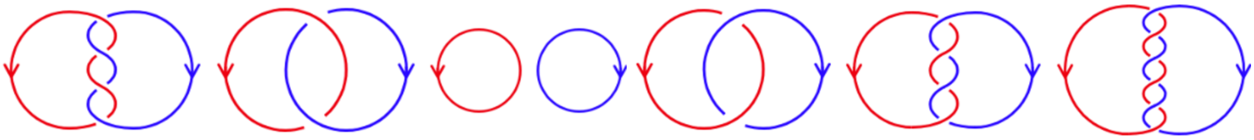
For any two knot K and J, we can think $K \cup J$ be the two-component link. Since J is a simple closed curve in $\mathbb{R}^3 - K$, we can think the equivalent class $[J]$ as a well-defined element in $\pi_1(\mathbb{R}^3 - K)$. Let t be the generator of $\pi_1(\mathbb{R}^3 - K)_{ab} \cong \mathbb{Z}$. Then, $[J] = t^k$ for some k.

Definition 2.3. Let K and J be the knots. A **linking number** $lk(K, J)$ of $K \cup J$ is defined as $lk(K, J) = k$, where $[J] = t^k$ and t is the generator of $\pi_1(\mathbb{R}^3 - K)_{ab} \cong \mathbb{Z}$,

For example, let's think about the case $K = \{x^2 + y^2 = 1\}$ and $J = \{(x-3)^2 + y^2 = 1\}$. If we consider $\pi_1(\mathbb{R}^3 - K)$ with the loops based at (3, 0, 0), J is homotopic to the constant loop. i.e, $[J] = 1 = t^0 \in \pi_1(\mathbb{R}^3 - K)$, Therefore, by definition, the linking number of $K \cup J$ is 0.



Let's think about this example, which is called the 'Hopf link'. Let K be the left knot and J be the right knot. If we choose the generator t of $\pi_1(\mathbb{R}^3 - K)_{ab}$ using the right hand rule, $[J] = t^{-1} \in \pi_1(\mathbb{R}^3 - K)_{ab}$. Therefore, $lk(K, J)$ = -1.



Using the similar approach, we could calculate the linking number of many 2-component link in the above figure. Starting from the left, the linking numbers are -2, -1, 0, 1, 2, 3 respectively.

Theorem 2.4. For a knot K, denote K with reversed orientation by K^r. Then $lk(K, J^r) = -lk(K, J) = lk(K^r, J)$

Proof. Let $lk(K, J) = k$ and t be the generator of $\pi_1(\mathbb{R}^3 - K)_{ab}$. Then, $[J] = t^k$ by definition. Since the generator induced by the reversed orientation is the inverse of the original generator, $[J] = t^{-1\times(-k)}$ in $\pi_1(\mathbb{R}^3 - K^r)_{ab}$. Therefore, $lk(K^r, J) = -k = -lk(K, J)$. Moreover, $[J^r] = [J]^{-1}$ in $\pi_1(\mathbb{R}^3 - K)_{ab}$ since $[J][J^r] = [J \cdot J^r] = 1$. Therefore, $[J^r] = [J]^{-1} = t^{-k}$, so $lk(K, J^r) = -k = -lk(K, J)$.

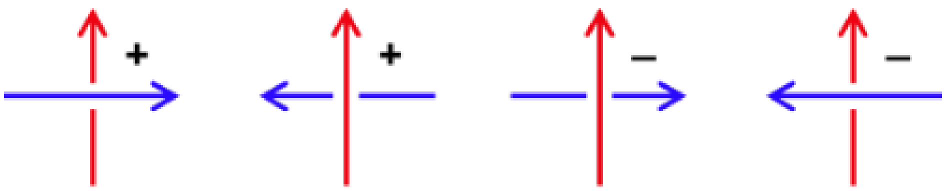
□

Definition 2.5. The **sign** of a crossing c in a link diagram is defined to be $sign\ c =$ $+1$(resp, $sign\ c = -1$) it it looks like 

The following theorem shows how to calculate the linking number of a link easily by the sign from its diagram.

Theorem 2.6. $lk(K, J) = \frac{1}{2}\sum_c sign\ c$, where c varies over crossings between K and J. Self-crossings of K and self-crossings of J are ignored.

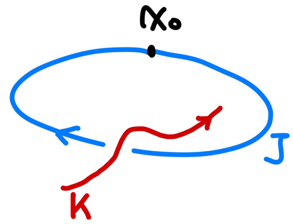
Proof. Let the red line as a strand of K and blue line as a strand of J, every crossing in a diagram is one of the cases below.



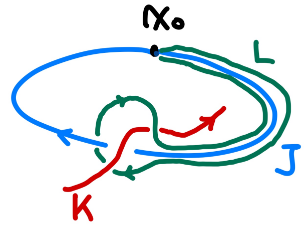
Denote the each case from 1 to 4. And let n_i be the number of case i in the link diagram. Then, $n_1 + n_2 - n_3 - n_4 = \sum_c sign\ c$.

If every crossing of a diagram is case 1 or 4, K is always in the behind of J, which means those two are unlinked. Then the linking number will be 0 in this case.

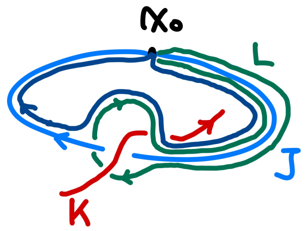
Now, suppose there are only one crossing of case 2, and other crossings are case 1 and 4. If we fix a basepoint x_0 of $\pi_1(\mathbb{R}^3 - K)$ on the J, J itself becomes the loop based at x_0. The situation will be like below.



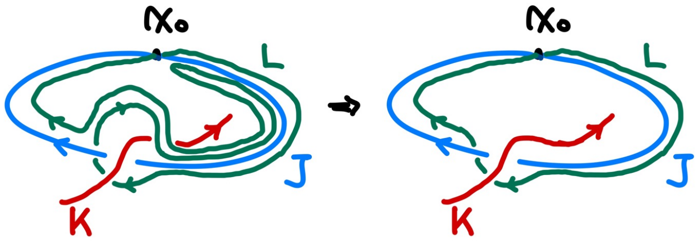
Then, let's think about another loop L based at x_0 as the figure below.



Then, if we multiply L with the new loop with deep blue color in the figure below,



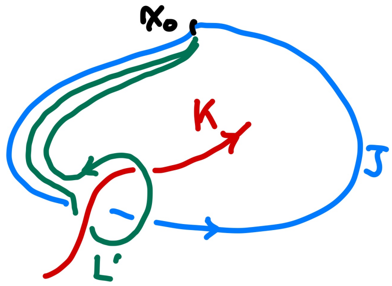
It is same with the original loop J, as we can see



Note that the deep blue loop and K are unlinked. So,

$[J] = [\text{Deep blue loop} \cdot L] = [\text{Deep blue loop}][L] = 1 \cdot [L] = [L]$ in $\pi_1(\mathbb{R}^3 - K)_{ab}$

This could be applied if there are multiple crossing of case 2. Also, crossing of case 3 can be figured out in the same way. Let L' be the loop made in case 3 which roles same with L in case 2, like the figure below.

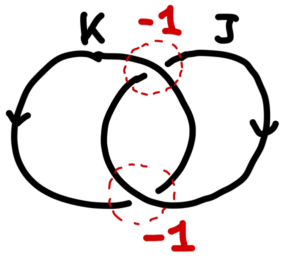


Then, if we do the same procedure for every crossing point of case 2 and case 4, n_2 equals to the number of such loops like L, and n_3 equals the number of such loops like L'. If we let t be the generator of $\pi_1(\mathbb{R}^3 - K)_{ab}$, then $[L] = t$ and $[L'] = t^{-1}$. Therefore, $[J] = [L]^{n_2}[L']^{n_3} = t^{n_2}t^{-n_3} = t^{n_2-n_3}$. Which means $lk(K, J) = n_2 - n_3$.

Similarly, if every crossing is case 2 and 3, K and J are unlinked. If we do the same procedure starting from that case, then we get $lk(K, J) = n_1 - n_4$. So, $lk(K, J) = n_1 - n_4 = n_2 - n_3$, therefore $lk(K, J) = \frac{1}{2}(n_1 + n_2 - n_3 - n_4) = \frac{1}{2}\sum_c sign\ c$

□

Now, if we revisit the example of a Hopf link, we could check that the theorem actually works well since $\frac{-1-1}{2} = -1$.



Theorem 2.7. $lk(K, J) = lk(J, K)$

Proof. Let's denote the number of case 1-4 crossing as $n_1 \cdots n_4$, in $lk(K, J)$, and denote the number of case 1 - 4 crossing as $n'_1 \cdots n'_4$ in $lk(J, K)$. Then, $n_1 = n'_2$ and $n_2 = n'_1$ since nothing changes but the order of who comes first change. Also, $n_3 = n'_4$ and $n_4 = n'_3$. Therefore, $lk(K, J) = \frac{1}{2}(n_1 + n_2 - n_3 - n_4) = \frac{1}{2}(n'_2 + n'_1 - n'_4 - n'_3) = lk(J, K)$.

□

Definition 2.8. Two links $K \cup J$ and $K' \cup J'$ in $\mathbb{R}^3$ are **equivalent** if and only if there exists a orientation-preserving homeomorphism $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\phi(K \cup J) = K' \cup J'$

Theorem 2.9. If two links $K \cup J$ and $K' \cup J'$ are equivalent, then $lk(K, J) = lk(K', J')$

Proof. Since $K \cup J$ and $K' \cup J'$ are equivalent, there exists an orientation-preserving homeomorphism $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\phi(K \cup J) = K' \cup J'$. If we restrict this homeomorphism ϕ to $\phi' : \mathbb{R}^3 - K \to \mathbb{R}^3 - K'$, it is still an orientation-preserving homeomorphism and satisfies $\phi'(J) = J'$. Then, this homeomorphism induces an isomorphism $\pi_1(\phi') : \pi_1(\mathbb{R}^3 - K) \to \pi_1(\mathbb{R}^3 - K')$. If we let t, t' be the generators of $\pi_1(\mathbb{R}^3 - K)_{ab}$ and $\pi_1(\mathbb{R}^3 - K')_{ab}$ respectively, $\pi_1(\phi')(t) = t'$ since ϕ' preserves the orientation.

Let $[J] = t^k \in \pi_1(\mathbb{R}^3 - K)_{ab}$ where $k = lk(K, J)$. Then, $\pi_1(\phi')([J]) = \pi_1(\phi')(t^k) = (\pi_1(\phi')(t))^k = (t')^k$. Since $\pi_1(\phi')([J]) = [J']$, we conclude that $[J'] = (t')^k$, which means $lk(K, J) = k = lk(K', J')$.

□

This theorem shows that the linking number is an invariant, which means we could distinguish whether two links are equivalent or not by computing the linking number.

3 Differential definition of the linking number

Before discussing about linking numbers, let's review what we have learned in the class. Let $M \subset \mathbb{R}^k$ and $N \subset \mathbb{R}^\ell$ be the smooth manifolds. Then, $M \times N \subset \mathbb{R}^{k+\ell}$ is a smooth manifold. Let M and N have dimension m and n respectively. When M and N are oriented by bases $(b_1, b_2, \cdots, b_m)$ and $(b'_1, b'_2 \cdots b'_n)$ of TM_x and TN_y, then $M \times N$ is oriented by the basis $(b_1, b_2, \cdots, b_m, b'_1, b'_2 \cdots b'_n)$ of $T(M \times N)_{(x,y)} = TM_x \times TN_y$.

Now, suppose $K \cup J$ is a 2-component link. Let's define a map $\Phi_{K,J} : K \times J \to S^2$ as follows :

$$\Phi_{K,J}(x, y) = \frac{x-y}{||x-y||}$$

Definition 3.1. $\ell(K, J) = deg\ \Phi_{K,J}$

Note that K and J are compact smooth manifolds of dimension 1. Moreover, $K \times J$ is oriented as a product as above, and S^2 is oriented as the boundary of D^3

Theorem 3.2. $\ell(K, J) = \ell(J, K)$

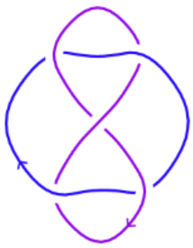
Proof. Let K and J be oriented by the basis $\{b\}$ and $\{b'\}$ of TK_x and TJ_y respectively. Then, $K \times J$ is oriented by the basis $\{b, b'\}$ of $TK_x \times TJ_y$, and $J \times K$ is oriented by the basis $\{b', b\}$ of $TJ_y \times TK_x$. Then, a map $\phi : K \times J \to J \times K$ defined by $h(x, y) = (y, x)$ is an orientation-reversing homeomorphism since $K \times J$ and $J \times K$ have different orientation. Since $\Phi_{K,J} = -\Phi_{J,K} \cdot \phi$, $deg\ \Phi_{K,J} = -deg\ (\Phi_{J,K} \cdot \phi) = -deg\ \Phi_{J,K} \cdot deg\ \phi = -deg\ \Phi_{J,K} \times -1 = deg\ \Phi_{J,K}$. Therefore $\ell(K,J) = deg\ \Phi_{K,J} = deg\ \Phi_{J,K} = \ell(K,J)$. $\square$

Theorem 3.3. Suppose there is an oriented surface Σ in $\mathbb{R}^3 - K$ such that $\partial\Sigma = J$. Then $\ell(K,J) = 0$

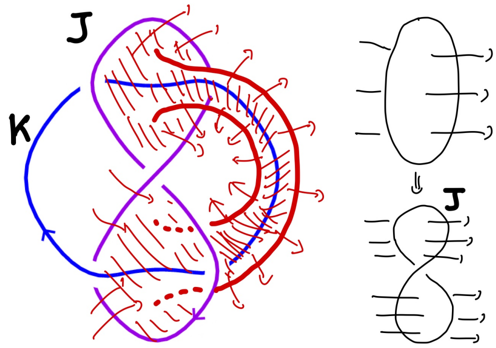
Proof. Since Σ and K are disjoint, we can think about some extended map $\Phi'_{K,\Sigma} : K \times \Sigma \to S^2$ as $\Phi'_{K,\Sigma}(x,y) = \frac{x-y}{||x-y||}$, which is a smooth extension of Φ. Since K is a 1-dimension manifold without boundary, $\partial(K \times \Sigma) \cong K \times \partial\Sigma = K \times J$. Now, $\Phi_{K,J} : K \times J \to S^2$ is smooth on the boundary $K \times J = \partial(K \times \Sigma)$. This can be smoothly extended to $K \times \Sigma$. Therefore, this implies that $deg\ \Phi_{K,J} = \ell(K,J) = 0$.

$\square$

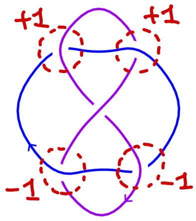
Let's apply this theorem to the example below.



If we think K be a blue knot and J be a purple knot, then we can attach an oriented surface Σ to J like the figure below.



So, by theorem 3.3, $\ell(K,J) = 0$. Note that $lk(K,J)$, defined in section 2 is also 0, which can be easily calculated by theorem 2.6. This is not a coincidence.



Theorem 3.4. $lk(K, J) = \ell(K, J)$

Proof. Let $(x, y, z) \in K$ and $(a, b, c) \in J$. Then,

$$\Phi_{K,J}((x, y, z), (a, b, c)) = \frac{(x-a,\ y-b,\ z-c)}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}$$

If we fix (a, b, c) and differentiate for (x, y, z), then

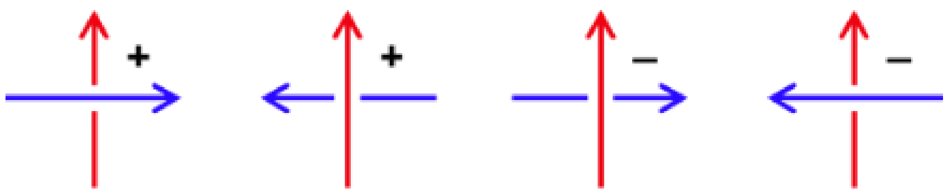
$$d\Phi_{K,J} = \frac{1}{((x-a)^2+(y-b)^2+(z-c)^2)^{(3/2)}} \begin{bmatrix} 1+(x-a)^2 & (x-a)(y-b) & (z-c)(x-a) \\ (x-a)(y-b) & 1+(y-b)^2 & (y-b)(z-c) \\ (z-c)(x-a) & (y-b)(z-c) & 1+(z-c)^2 \end{bmatrix}$$

Then, the determinant becomes

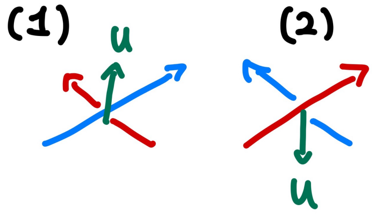
$$\det(d\Phi_{K,J}) = \frac{1+(x-a)^2+(y-b)^2+(z-c)^2}{((x-a)^2+(y-b)^2+(z-c)^2)^{(3/2)}}$$

Since the determinant is always positive, there is no critical point. By the similar way, we could get the same result when we fix (x, y, z) and differentiate for (a, b, c). Since $TM(X \times Y)_{x,y} \cong TM_x \times TM_y$, $\Phi : K \times J \to S^2$ has no critical point.

Now, suppose K is the blue strand and J is the red strand in the figure below. As we discussed in the theorem 2.6, there are four cases of crossing in the diagram of $K \cup J$.



We can think a link diagram as an orthogonal projection of the link to the plane perpendicular to unit vector $u \in S^2$. Then, u is a regular value since $\Phi : K \times J \to S^2$ has no critical point. Also, if we fix u as a direction of going down to the plane, the preimages of u are from the second and third case in the crossing, where J is higher than K.

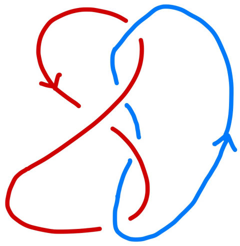


First, let's consider the case 2. Let (α, β) be the preimage of u, which are the points on K and J respectively. Let $\{t\}$ and $\{s\}$ be the basis of TK_α and TJ_β. Then, $\{t, s\}$ is an oriented basis of $T(K \times J)_{\alpha,\beta}$. Then, $d\Phi$ locally preserves the orientation near (α, β) by the right hand rule. So, $deg\ d\Phi_{(\alpha,\beta)} = 1$. Similarly, for the third case, $deg\ d\Phi_{(\alpha,\beta)} = -1$ since the basis of TM_β is reversed, so $d\Phi_{(\alpha,\beta)}$ reverses the orientation.

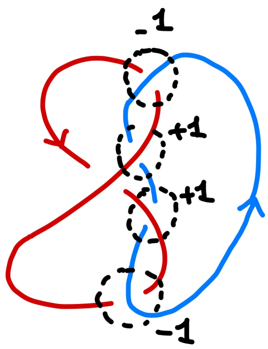
If we let n_2 and n_3 be the number of second and third case of crossing, then $\ell(K, J) = deg\ \Phi_{K,J} = n_2 - n_3 = lk(K, J)$ as we mentioned in the proof of theorem 2.6. $\square$

So, the algebraic and differential approach for linking number were actually same!

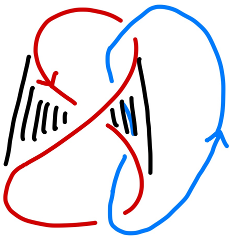
Note that the converse of the theorem 3.3 is not be true. i.e, some 2-component link with $\ell(K, J) = 0$ might not have an oriented surface whose boundary is J. Here's an example.



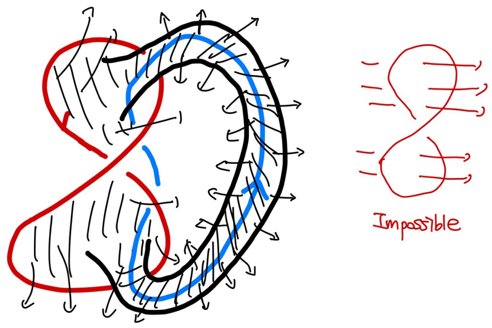
By theorem 2.6, we could easily check that $lk(K, J) = \ell(K, J) = 0$



However, if we think blue knot as J and red knot as K, there are two possibilities to attach an oriented surface with boundary J : filling the space between circle, or constructing a new surface along J. The first possibility is like below.



But filling the space in this way needs a Mobius band, which is not oriented. The second possibility is like below, which is impossible.



So, the converse of theorem 3.3 is generally not true.