A glimpse of algebraic combinatorics

Seokhwan Moon

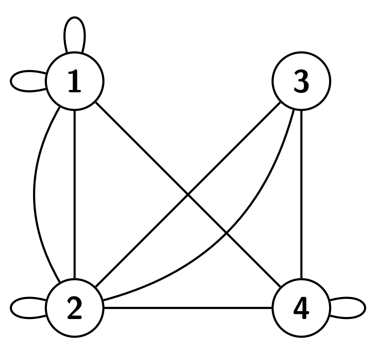
May 29, 2023

In combinatorics, especially on graph theory, walk is one of the main research subject. Combinatorial approaches about the subject are quite useful, but algebraic approach gives us some new ideas. The goal of this project is to describe walks in graph by using algebraic language. Concepts in linear algebra such as basis, eigenvalue, eigenvector, orthogonal matrix, diagonalize, and their properties are required.

1 Walks in graph

Definition 1. *For a finite undirected graph G with vertex $V = \{v_1, v_2, \cdots v_p\}$ and edge $E = \{e_1, e_2, \cdots, e_q\}$, an adjacency matrix $A(G)$ of a graph G is a $p \times p$ matrix such that (i, j)-th entry is equal to the number of edges connecting two vertex v_i and v_j.*

For example, if we have a graph G as a figure below,



we could get the adjacency matrix as

$$A(G) = \begin{bmatrix} 2 & 2 & 0 & 1 \\ 2 & 1 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Definition 2. *A **walk** in a graph G of length ℓ from vertex u to v is a sequence $v_1 = u$, e_1, v_2, e_2, v_3 $\cdots$, v_ℓ, e_ℓ, $v_{\ell+1} = v$ such that each e_i is a vertex connecting v_i and v_{i+1}.*

Let's observe the matrix $A(G)$. When we multiply this matrix ℓ times, each (i, j)-th entry will be

$$(A(G)^\ell)_{ij} = \sum a_{ik_1} a_{k_1 k_2} \cdots a_{k_{l-1} j}$$

What does this mean? each $k_1, k_2, \cdots, k_{\ell-1}$ denote one of the vertices in a graph, and $a_{k_m k_n}$ means the number of edges connecting vertex v_{k_m} and v_{k_n}. So, we could observe that the (i, j)-th entry of the matrix $A(G)^l$ means the number of walks from v_i to v_j with length ℓ

Obviously, $A(G)$ is a $p \times p$ real symmetric matrix for any graph G, since the entry a_{ij} and a_{ji} are equal by definition. Therefore, $A(G)$ has real eigenvalues, and has p independent real eigenvectors, and also diagonalizable. If we choose such eigenvector as orthonormal vectors, the matrix generated by those eigenvectors will be an orthogonal matrix.

Denote u_1, u_2, $\cdots$, u_p be the real orthonormal eigenvectors of $A(G)$ with corresponding eigenvalues λ_1, λ_2, $\cdots$, λ_p. Then the matrix $U = [u_1\ u_2\ \cdots\ u_p]$ is an orthogonal matrix. i.e, $U^T = U^{-1}$. Thus, U diagonalizes $A(G)$ as

$$U^{-1} A(G) U = diag(\lambda_1, \lambda_2, \cdots, \lambda_p)$$

Theorem 1.1. *For any integer $\ell \geq 1$, the (i, j)-th entry of a matrix $A(G)^\ell$ is equal to the number of walks from v_i to v_j with length ℓ. Moreover, if we denote $U = (u_{ij})$,*

$$(A(G)^\ell)_{ij} = \sum_{k=1}^{p} u_{ik} \lambda_k^\ell u_{jk}$$

Proof. It is obvious from the observation that the (i, j)-th entry of a matrix $A(G)^\ell$ is equal to the number of walks from v_i to v_j with length ℓ. Moreover,

$$U^{-1} A(G)^\ell U = diag(\lambda_1 \ell, \lambda_2^\ell, \cdots, \lambda_p^\ell).$$

Therefore,

$$A(G)^\ell = U diag(\lambda_1^\ell, \lambda_2^\ell, \cdots, \lambda_p^\ell) U^{-1}$$

which directly shows our conclusion. Note that the latter matrix in the upper equation is $U^{-1} = U^T$, so $u_{kj}^{-1} = u_{jk}$ $\square$

Now, a **closed walk** is a walk that ends where it begins. It is straightforward that the number of closed walk with length ℓ in G is equal to

$$\sum_{i=1}^{p}(A(G)^{\ell})_{ii} = tr(A(G)^{\ell})$$

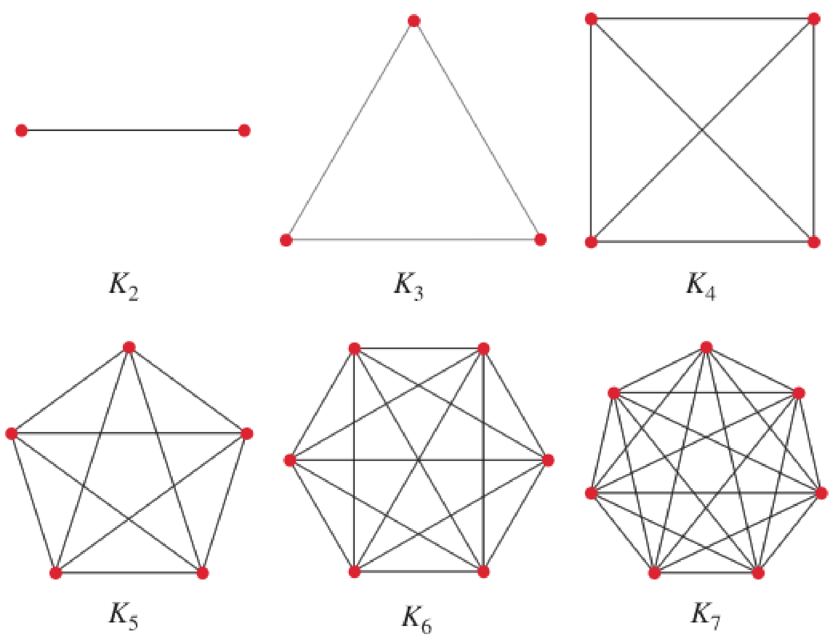
Since the trace of square matrix is equal to the sum of eigenvalues, we could conclude the following.

Theorem 1.2. *If $A(G)$ has eigenvalues $\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_p$, the number of closed walks in G of length ℓ is equal to*

$$\lambda_1^{\ell} + \lambda_2^{\ell} + \cdots + \lambda_p^{\ell}$$

Now, let's apply this theorem to some examples. The most simple example is a complete graph.

Definition 3. *A **complete graph** K_p is a graph with vertex $V = \{v_1, v_2, \cdots v_p\}$ and there exists only one edge between any two distinct vetrices.*



Applying the above theorem, we could conclude the following.

Theorem 1.3. *The number of closed walks of length ℓ in a complete graph K_p starting and ends at a vertex i is*

$$(A(K_p)^\ell)_{ii} = \frac{1}{p}((p-1)^\ell + (p-1)(-1)^\ell)$$

Proof. The adjacency matrix $A(K_p)$ is equal to $J - I$, where J is a $p \times p$ matrix with all entry 1, and I is an identity $p \times p$ matrix.

Since the eigenvalue of J is p (with multiplicity 1) and 0 (with multiplicity $p-1$), the eigenvalues of $A(K_p) = J - I$ is $p-1$ (with multiplicity 1) and -1 (with multiplicity $p-1$). Applying Theorem 1.2 gives the number of total closed walks of length ℓ in G. Therefore, by symmetry, dividing with p concludes the result. □

The reference book mentioned that the combinatorial proof is quite tricky. But actually it is quite simple. Here's an alternative proof of the Theorem 1.3

Proof. Let's think that some people is moving along a closed walk. If the closed walk of length ℓ starts and ends at vertex i, it means that the people who was initially at vertex i moved ℓ times and come back to i.

To be at vertex i after ℓth movement, that person could be at any vertex except i after $l-1$th movement. So, if we denote $A(\ell)$ as the number of closed walks of length ℓ in a complete graph K_p starting from vertex i,

$$A(\ell) = (p-1)^{l-1} - A(\ell - 1)$$

since the number of possible walk of length $\ell - 1$ on K_p is $(p-1)^{\ell-1}$, by choosing which vertex to go among all vertex except the vertex where the person is in at each movement.

If ℓ is even,

$$\begin{aligned}
A(\ell) + A(\ell-1) &= (p-1)^{\ell-1} \\
-A(\ell-1) - A(\ell-2) &= -(p-1)^{\ell-2} \\
A(\ell-2) + A(\ell-3) &= (p-1)^{\ell-3} \\
-A(\ell-3) - A(\ell-4) &= -(p-1)^{\ell-4} \\
&\vdots \\
-A(3) - A(2) &= -(p-1)^2 \\
A(2) + A(1) &= (p-1)^1
\end{aligned}$$

Summing up, we get

$$A(\ell) + A(1) = \sum_{k=1}^{\ell-1} (-1)^{k+1}(p-1)^k = \frac{1}{p}((p-1)^\ell + (p-1))$$

Because there is no loop in a complete graph, $A(1)$=0. Similarly, if ℓ is odd,

$$\begin{aligned} A(\ell) + A(\ell-1) &= (p-1)^{\ell-1} \\ -A(\ell-1) - A(\ell-2) &= -(p-1)^{\ell-2} \\ A(\ell-2) + A(\ell-3) &= (p-1)^{\ell-3} \\ -A(\ell-3) - A(\ell-4) &= -(p-1)^{\ell-4} \\ &\vdots \\ A(3) + A(2) &= (p-1)^2 \\ -A(2) - A(1) &= -(p-1)^1 \end{aligned}$$

Summing up, we get

$$A(\ell) - A(1) = A(l) = \sum_{k=1}^{\ell-1} (-1)^k (p-1)^k = \frac{1}{p}((p-1)^\ell - (p-1))$$

□

How about non-closed walk? Since I and J commute, using (generalized) binomial theorem, we could denote $A(K_p)^\ell$ as follows :

$$\begin{aligned} A(K_p)^\ell = (J-I)^\ell &= \sum_{k=0}^{\ell} \binom{\ell}{k} J^k (-I)^{\ell-k} \\ &= \sum_{k=0}^{\ell} (-1)^{\ell-k} \binom{\ell}{k} p^{k-1} J \\ &= (-1)^\ell I + \sum_{k=1}^{\ell} (-1)^{\ell-k} \binom{\ell}{k} p^{k-1} J \\ &= \frac{1}{p}((p-1)^\ell - (-1)^\ell) J + (-1)^\ell I \end{aligned}$$

Therefore, if $i \neq j$,

$$(A(K_p)^\ell)_{ij} = (J-I)^\ell_{ij} = \frac{1}{p}((p-1)^\ell - (-1)^\ell)$$

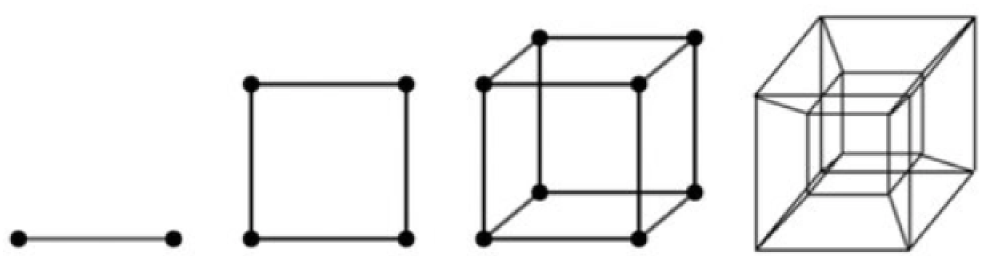
Of course,

$$\begin{aligned}(A(K_p)^\ell)_{ii} = (J-I)^\ell_{ii} &= \frac{1}{p}((p-1)^\ell - (-1)^\ell) + (-1)^\ell \\ &= \frac{1}{p}((p-1)^\ell + (p-1)(-1)^\ell)\end{aligned}$$

2 Cubes and the Radon Transformation

Now, let's think about more complicated graph.

Definition 4. *A* ***n-cube*** *is a graph C_n with vertex $V(C_n) = \mathbb{Z}_2^n$ and two vertex are connected by an edge if they differ in exactly one component.*



Of course, we could directly apply the theorem in the former section, but let's use some algebraic technique. Let $\mathcal{V} = \{f : \mathbb{Z}_2^n \to \mathbb{R}\}$, which is a vector space over $\mathbb{R}$ of dimension 2^n. (You could easily check conditions of being a vector space) Now, let's think some basis of this vector space.

For each $u \in V(C_n) = \mathbb{Z}_2^n$,

Basis 1 :

$$f_u(v) = \delta_{uv} = \begin{cases} 1 & \text{if } u = v \\ 0 & \text{otherwise} \end{cases}$$

Basis 2 :

$$\chi_u(v) = (-1)^{u \cdot v}$$

It is easy to check that those two are the basis. (Note that for any $f, g \in \mathcal{V}$, $< f, g >= \sum_{u \in \mathbb{Z}_2^n} f(u)g(u))$ Actually, basis 1 is somewhat we could normally imagine. i.e. for any $g \in \mathcal{V}$,

$$g = \sum_{u \in \mathbb{Z}_2^n} g(u) f_u$$

since the domain of g is discrete and finite. However, it is no easy to directly imagine how does the basis 2 actually look like. So, what we are gonna do is to find some linear transformation between those two basis.

Definition 5. *For a subset $\Gamma \subset \mathbb{Z}_2^n$ and a function $f \in \mathcal{V}$, the* ***Radon Transformation*** *of f is a function $\Phi_\Gamma f : \mathcal{V} \to \mathcal{V}$*

$$\Phi_\Gamma f(v) = \sum_{w \in \Gamma} f(v + w)$$

It is almost obvious from the definition that the Radon transformation is a linear transformation, so it could be represented as a matrix. Then, we have the following theorem (as we wanted).

Theorem 2.1. *χ_u is the eigenvectors of Φ_Γ. Moreover, the corresponding eigenvalues λ_u such that $\Phi_\Gamma \chi_u = \lambda_u \chi_u$ is given by*

$$\lambda_u = \sum_{w \in \Gamma} (-1)^{u \cdot w}$$

Proof.

$$\begin{aligned}
\Phi_\Gamma \chi_u(v) &= \sum_{w \in \Gamma} \chi_u(v + w) \\
&= \sum_{w \in \Gamma} (-1)^{u \cdot (v+w)} \\
&= (\sum_{w \in \Gamma} (-1)^{u \cdot w})(-1)^{u \cdot v} \\
&= (\sum_{w \in \Gamma} (-1)^{u \cdot w}) \chi_u(v)
\end{aligned}$$

□

Now, let $\Delta = \{\delta_1, \delta_2, \cdots, \delta_n\}$ such that $\delta_i = (\delta_{i1}, \delta_{i2}, \cdots, \delta_{in})$. Then, $\Phi_\Delta f_u : V \to V$ is a linear transformation, and let $[\Phi_\Delta]$ be the matrix representation with respect to the Basis 1.

Theorem 2.2. $[\Phi_\Delta] = A(C_n)$

Proof. Let $v \in \mathbb{Z}_2^n$. Since $u = v + w$ iff $u + w = v$ in $\mathbb{Z}_2^n$,

$$\begin{aligned}\Phi_\Delta f_u(v) &= \sum_{w \in \Delta} f_u(v + w) \\ &= \sum_{w \in \Delta} f_{u+w}(v)\end{aligned}$$

so,

$$\Phi_\Delta f_u = \sum_{w \in \Delta} f_{u+w}$$

Note that $u + v \in \Delta$ iff u and v are different in exactly one coordinate, which is exactly the condition to be an edge of C_n. Moreover,

$$\begin{aligned}[\Phi_\Delta]_{u,v} = 1 &\leftrightarrow \Phi_\Delta f_u = \cdots + 1 \cdot f_v + \cdot \\ &\leftrightarrow u + w = v \textit{ and } w \in \Delta \\ &\leftrightarrow u + v \in \Delta\end{aligned}$$

So, we get

$$[\Phi_\Delta]_{u,v} = \begin{cases} 1 & \text{if } u + v \in \Delta \\ 0 & \text{otherwise} \end{cases}$$

□

This theorem means that the matrix representation of Φ_Δ with respect to the Basis 1 is equal to the matrix representation of $A(C_n)$ with respect to the vertices of $V(C_n) = \mathbb{Z}_2^n$. So, by using above tho theorems, we could get the following. (I'm not gonna prove it rigorously, because this is a **Glimpse** in algebraic combinatorics, not a linear algebra class)

Theorem 2.3. *For each $u \in \mathbb{Z}_2^n$, the eigenvectors E_u of $A(C_n)$ are given by*

$$E_u = \sum_{v \in \mathbb{Z}_2^n} (-1)^{u \cdot v} v$$

and the corresponding eigenvalue λ_u is

$$\lambda_u = n - 2w(u)$$

where $w(u)$ is the number of 1's in u. In other words, $A(C_n)$ has eigenvalue $n - 2i$ with multiplicity $\binom{n}{i}$ for each $0 \le i \le n$

Now, we have eigenvalues, eigenvectors, so we found everything to use theorem 1.1

Theorem 2.4. *Let $u, v \in \mathbb{Z}_2^n$ and $w(u+v) = k$. Then the number of walks of length l in C_n between u and v is given as*

$$(A^l)_{uv} = \frac{1}{2^n}\sum_{i=0}^{n}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\binom{n-k}{i-j}(n-2i)^l$$

The reference book mentioned that the combinatorial proof of this theorem is possible, but very long and hard. Instead, let's find some combinatorial proof of some special case :

$$(A^l)_{uu} = \frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}(n-2i)^l$$

Proof. On the n-cube, let's denote move as i if the movement increases the i-th coordinate by 1, $-i$ if the movement decreases the i-th coordinate by 1. Then, the set of all possible movements will be

$$1,\ -1,\ 2,\ -2,\ \cdots n,\ -n$$

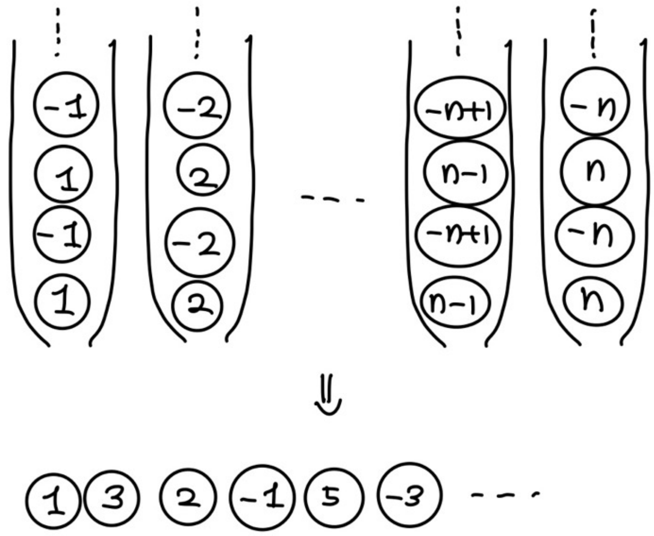
and the walk of length ℓ on n-cube can be written as a word of length ℓ generated by those number-letters. It is obvious that if i appeared in a word, i could not appear again until $-i$ appears.

How can we describe a closed walk? A closed walk is described as a word such that for each $i \in \{1,\ 2,\ \cdots n\}$, if we delete every letters which does not contain i, the left letters will be like

$$i,\ -i,\ i,\ -i,\ \cdots i,\ -i$$

Of course, starting from whether i or $-i$ depends on where we start. Without loss of generality, let's suppose the walk starts and ends from $(0, 0, \cdots, 0)$

Then, we can think a closed walk as a figure below



In the each box, balls are already ordered in such way we want. And we just choose which box to get the ball. Of course, the total number of balls from each box must be even. Suppose we picked $2k_i$ balls for each i. Then, the number of ordering is

$$\sum_{2k_1+2k_2+\cdots+2k_n=\ell} \frac{\ell!}{(2k_1)!(2k_2)!\cdots(2k_n)!} = \ell! \sum_{\sum 2k_i=\ell} \prod_{i=0}^{n} \frac{1}{(2k_i)!}$$

It is hard to calculate this number. Instead, let's express these numbers as a exponential generating function. Then the exponential generating function will be

$$\begin{aligned}
f(x) &= \sum_{\ell=0}^{\infty} \Big(\sum_{\sum 2k_i=\ell} \prod_{i=0}^{n} \frac{1}{(2k_i)!} \Big) x^{\ell} \\
&= \prod_{i=0}^{n} \sum_{k_i=0}^{\infty} \frac{x^{2k_i}}{(2k_i)!} \\
&= \Big(\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \Big)^n \\
&= \Big(\frac{e^x + e^{-x}}{2} \Big)^n = \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} e^{(n-2i)x}
\end{aligned}$$

So the number we want is the coefficient of $\frac{x^\ell}{\ell!}$.

$$\frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}e^{(n-2i)x} = \frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}\sum_{k=0}^{\infty}\frac{(n-2i)^k}{k!}x^k$$

which is shown as

$$\frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}(n-2i)^\ell$$

$\square$

Not a short proof, but the main idea is quite interesting.