Probabilistic proof of RSK correspondence

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Abstract

The Robinson–Schensted–Knuth (RSK) correspondence is a bijection between permutations in S_n and pairs of standard tableuax. This correspondence can be written as

$$|S_n| = \sum_{\lambda} f_{\lambda}^2$$

where f_λ denotes the number of Standard Young Tableau of shape λ with size n. There are several proofs for this correspondence. In this report, we will follow the probabilistic proof, which is done by Curtis Greene, Albert Nijenhuis, and Herbert S. Wilf in 1984 [1].

1 Definition and Notations

Let λ be the partition of n with $\lambda_1 \geq \lambda_2 \geq \cdots$. Then, the *conjugate* λ^* of λ is given by

$$\lambda_i^* = |\{j : \lambda_j \geq i\}| \tag{1}$$

which is the shape after reflecting λ along the diagonal.

The *hook* $H(a, b)$ of the cell (a, b) consists of all cells (u, v) such that either $u = a$ and $b \leq v \leq \lambda_a$ of $v = b$ and $a \leq u \leq \lambda_b^*$. The *hook length* h_{ab} is

$$h_{ab} = |H(a, b)| = (\lambda_a - b) + (\lambda_b^* - a) + 1 \tag{2}$$

The *hook walk* is as follows. Let λ be a partition of n. Starting from a cell (a, b), we choose the next cell in the walk uniformly from the other cells in the hook of previous cells. If the corner is reached, the walk is finished.

2 Theorem and the proof

Theorem 1 *(RSK Correspondence)*

$$|S_n| = n! = \sum_{\lambda} f_{\lambda}^2$$

where f_λ is the number of standard tableaux of shape λ with size n.

The motivation of the proof is as follows

- If we rewrite the equation,

$$\sum_{\lambda} \frac{f_{\lambda}^2}{n!} = 1 \tag{3}$$

we want to think $\frac{f_\lambda^2}{n!}$ as the *probability that the Young diagram with shape λ is generated by some procedure*. Then it is obvious that the summing up the probability for all $\lambda \vdash n$ is equal to 1.

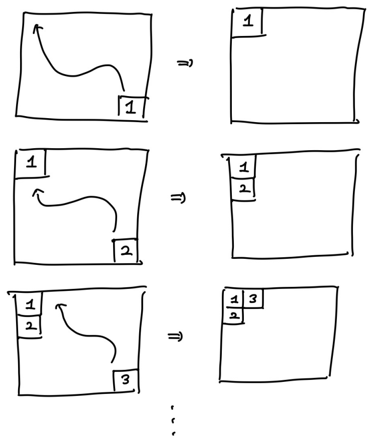

Figure 1: Procedure of generating the Standard Young Tableau

- Then what is the procedure that generates the Young diagram? One natural procedure that we can think is as following.

  On the right below corner of the blank board, we place new block. Then by some pathway, the block moves to the left upper direction until the corner is reached. Repeating the same procedure until the block marked with n is placed to the corner makes the Standard Young Tableau (SYT) whose shape is the partition of n.

- Then, what is the pathway that moves the block to the upper left corner? And how can we select the size of the board?

Before the proof, let's begin with the following lemmas.

Lemma 1 *(Hook length formula) Let f_λ be the number of standard tableaux of shape λ with size n. Then,*

$$f_\lambda = \frac{n!}{\prod_{(a,b)\in\lambda} h_{ab}} \tag{4}$$

Lemma 2 *Let $p(\alpha,\beta|a,b)$ be the probability of the hook walk starts at (a,b) and terminates at (α,β). Then,*

$$p(\alpha,\beta|a,b) = \prod_{a\le i<\alpha} \frac{h_{i+1,\beta}}{h_{i,\beta}-1} \prod_{b\le j<\beta} \frac{h_{\alpha,j+1}}{h_{\alpha,j}-1} \tag{5}$$

Proof. In class, we showed that

$$p(\alpha,\beta|a,b) = \{\frac{1}{h_{a\beta}} \prod_{a\le i<\alpha} \frac{h_{i\beta}}{h_{i\beta}-1}\}\{\frac{1}{h_{\alpha b}} \prod_{b\le j<\beta} \frac{h_{\alpha j}}{h_{\alpha j}-1}\} \tag{6}$$

provided that $\alpha\ge a$, $\beta\ge b$, $(\alpha,\beta)\neq(a,b)$. By observation, we get

$$p(\alpha,\beta|a,b) = p(\alpha,\beta|a,\beta)p(\alpha,\beta|\alpha,b) \tag{7}$$

By the definition of the hook walk, we can think as the following : the hook walk starting from (a,b) must reach one of the cells in the hook except itself. i.e., the first step of the hook walk will reach some cell among $h_{ab}-1$ number of cells, and we calculate the probability of the hook walk starting from that cell and ends at (α,β). This implies

$$p(\alpha,\beta|a,b) = \frac{1}{h_{ab}-1}\{\sum_{a<i\le\alpha} p(\alpha,\beta|i,b) + \sum_{b<j\le\beta} p(\alpha,\beta|a,j)\} \tag{8}$$

If $\alpha > a$ and $\beta = b$, this becomes

$$(h_{a\beta} - 1)p(\alpha, \beta|a, \beta) = \sum_{a<i\leq\alpha} p(\alpha, \beta|i, b) \tag{9}$$

Subtracting (9) the corresponding sum with a replaced by $a + 1$ gives

$$p(\alpha, \beta|a, \beta) = \frac{h_{a+1,\beta}}{h_{a\beta} - 1} p(\alpha, \beta|a + 1, \beta) \tag{10}$$

so, inductively, we get

$$p(\alpha, \beta|a, \beta) = \prod_{a\leq i<\alpha} \frac{h_{i+1,\beta}}{h_{i\beta} - 1} \tag{11}$$

similarly, starting from the case where $\alpha = a$ and $\beta > b$, we get

$$(h_{\alpha b} - 1)p(\alpha, \beta|\alpha, b) = \sum_{b<j\leq\beta} p(\alpha, \beta|a, j) \tag{12}$$

and

$$p(\alpha, \beta|\alpha, b) = \prod_{b\leq j<\beta} \frac{h_{\alpha,j+1}}{h_{\alpha j} - 1} \tag{13}$$

Plugging (11) and (13) to (7) completes the proof.

3 Complementary boards and special complementary hook walk

Definition 1 *(Complementary board) Let λ be a Young diagram which is the partition of integer n. Select two integers p, q such that $p \geq \lambda_1^*$, $q \geq \lambda_1$. Then a* ***complementary board*** *λ' of λ is*

$$\lambda' = \{(x, y) : 0 < q + 1 - y \leq \lambda_{p+1-x}, 0 < x \leq p\}$$

i.e., S/λ where S is the regular board with size $p \times q$

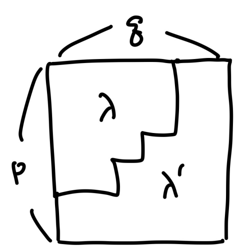

Figure 2: Young diagram λ and its complementary board λ'

The *special complementary hook walk* is the hook walk on complementary board starting from (p, q) and moves left and up. i.e., the procedure is as follows

- Choose a complementary board λ' of λ.
- Start from the cell (p, q).
- Choose a next cell uniformly from the complementary hook of previous cell except itself.
- The walk ends when it is reached at the corner.

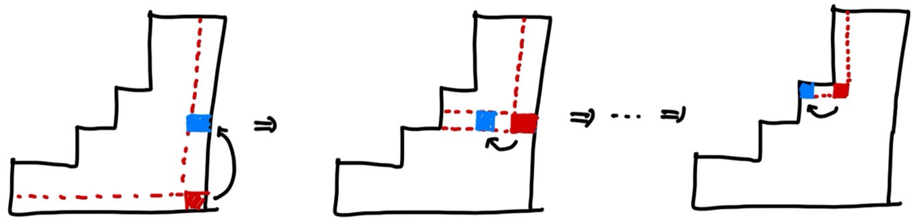

Figure 3: Process of special complementary hook walk. Red cell denotes the previous position, and blue cell denotes the position after the movement.

Definition 2 *The **distance** between cells (u, v) and (u', v') is defined as*

$$d((u, v), (u', v')) = |u - u'| + |v - v'| \tag{14}$$

Proposition 1 *Let λ be the Young diagram which is the partition of n, and let p, q such that $p > \lambda_1^*$, $q > \lambda_1$. Then the probability that a special complementary hook walk in the complementary board will terminate at the corner $\bar{\kappa}$ of T' equals*

$$\bar{p}(\bar{\kappa}) = \prod_{\sigma} d(\bar{\kappa}, \sigma) / \prod_{\bar{\sigma}} d(\bar{\kappa}, \bar{\sigma}) \tag{15}$$

where σ and $\bar{\sigma}$ ranges over all corner cells of T and T'. i.e., the probability that the special complementary hook walk terminate at some corner is independent from the choice of p, q.

Proof. Let's rotate the board 180 degree and think about (general) hook walk which starts at $(1, 1)$. Let $p(\alpha, \beta|1, 1)$ be the probability of the hook walk starts at $(1, 1)$ and terminates at (α, β). Then by Lemma 2,

$$p(\alpha, \beta|1, 1) = \prod_{1 \le i \le \alpha} \frac{h_{i+1,\beta}}{h_{i,\beta} - 1} \prod_{1 \le j \le \beta} \frac{h_{\alpha,j+1}}{h_{\alpha,j} - 1} \tag{16}$$

Looking at the factors in the product, we can see that the factor is not 1 only when the two endpoint of the hook of $(i + 1, \beta)$ corresponds to (α, β) and other corner cell. (Colored cells in the below figure)

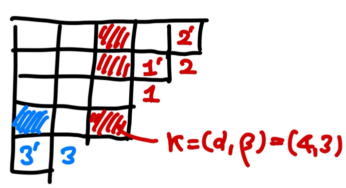

Figure 4: Toy example for the proof of Proposition 1

For this specific example, we get

$$p(\alpha, \beta|1, 1) = \prod_{1 \le i \le \alpha} \frac{h_{i+1,\beta}}{h_{i,\beta} - 1} \prod_{1 \le j \le \beta} \frac{h_{\alpha,j+1}}{h_{\alpha,j} - 1} \tag{17}$$

$$= \frac{h_{3,3}}{h_{2,3} - 1} \frac{h_{2,3}}{h_{1,3} - 1} \frac{h_{4,2}}{h_{4,1} - 1} \tag{18}$$

$$= \frac{d(1, \kappa)}{d(1', \kappa)} \frac{d(2, \kappa)}{d(2', \kappa)} \frac{d(3, \kappa)}{d(3', \kappa)} \tag{19}$$

We can notice that each hook length in the denominator corresponds to the length between κ and the corner cell of λ, and each hook length in the numerator corresponds to the length between κ and the corner cell of λ'. Transposing the observation to the complementary board with appropriate change yields the conclusion.

Proposition 2 *Let T be the SYT with shape λ, and T' be the SYT with shape λ' obtained from λ by adjoining a cell $\bar{\kappa} = (\alpha, \beta)$. Then,*

$$\bar{p}(\bar{\kappa}) = \frac{f_{\lambda'}}{n f_\lambda} \tag{20}$$

where $\lambda \vdash n$.

Proof. Using lemma 1, we get

$$\frac{f_{\lambda'}}{n f_\lambda} = \frac{n!}{\prod_{(a,b)\in\lambda} h_{ab}} \frac{1}{n} \frac{\prod_{(a,b)\in\lambda'} h_{ab}}{(n-1)!} \tag{21}$$

$$= \frac{\prod_{(a,b)\in\lambda'} h_{ab}}{\prod_{(a,b)\in\lambda} h_{ab}} \tag{22}$$

Note that the hook length for λ and λ' is exactly same except the cells (a, b) where $a = \alpha$ or $b = \beta$. Also, for those cases, we could notice that the hook length in λ is one less than the hook length in λ'. Therefore, we get

$$\frac{f_{\lambda'}}{n f_\lambda} = \frac{\prod_{(a,b)\in\lambda'} h_{ab}}{\prod_{(a,b)\in\lambda} h_{ab}} \tag{23}$$

$$= \prod_{1\le i\le\alpha} \frac{h_{i+1,\beta}}{h_{i,\beta} - 1} \prod_{1\le j\le\beta} \frac{h_{\alpha,j+1}}{h_{\alpha,j} - 1} \tag{24}$$

which is exactly same with (17), so we get the conclusion.

4 Proof of the theorem

Proposition 3 *Let n, p, q be the positive integers such that $p > n$ and $q > n$. Starting with an empty tableaux, a taleaux of size i $(1 \le i \le n)$ is constructed from a tableaux of size $i - 1$ be inserting i into the terminal cell of a special complementary hook walk in the complementary board consisting of those cells in the rectangular board with corner (p, q) which have not been terminal cells in prior walks. By this procedure, construct a SYT T with shape λ. Then the probability of T being produced is $f_\lambda/n!$.*

Proof. Using induction. If $|T| = 1$, then it is obvious. Then, suppose that the equation is satisfied for some T with $|T| = n$. Let's denote T' which is the SYT by adjoining the $n + 1$th cell to T. Then, the probability of T' to be generated is the product of probability that T generated and the probability that $n + 1$th cell is adjoined. i.e. using Proposition 2,

$$\frac{f_\lambda}{n!} \times \frac{f_{\lambda'}}{(n+1) f_\lambda} = \frac{f_{\lambda'}}{(n+1)!} \tag{25}$$

which satisfies the equation.

Theorem 2 *The probability of Young diagram with shape λ is generated is $\frac{f_\lambda^2}{n!}$. Therefore,*

$$\sum_{\lambda\vdash n} \frac{f_\lambda^2}{n!} = 1 \tag{26}$$

Proof. By Proposition 3, the probability of SYT T with shape λ being generated is $\frac{f_\lambda}{n!}$. Since there are f_λ number of SYT with shape λ, multiplying this values gives the conclusion.

5 Conclusion

In this report, we showed the probabilistic proof of the RSK correspondence. The main idea for the proof was to think $f_\lambda^2/n!$ as the probability that λ is generated. To specify the process generating

Young diagram, we construct some procedure that new cells begin moving from right below to the upper left direction so that the new cell adjoins to the SYT. By using hook length formula and special complementary hook walk, we proved that the probability that λ is generated is $f_\lambda^2/n!$.

The main idea for this proof that *generating a Young diagram by probabilistic procedure* is widely applicable. For example, in the reference paper [2], the authors made another process of generating a Young diagram, and proved the same formula by the similar procedure. I guess it is possible to apply similar idea for generating a plactic monoid [3], even though one should be careful for the Knuth equivalence. It would be interesting to further apply those ideas to other formulas.

References

[1] Curtis Greene, Albert Nijenhuis, and Herbert S. Wilf. Another probabilistic method in the theory of young tableaux. *Journal of Combinatorial Theory, Series A*, 37(2):127–135, 1984.

[2] B Pittel. On growing a random young tableau. *Journal of Combinatorial Theory, Series A*, 41(2):278–285, 1986.

[3] M. Lothaire. The plactic monoid. *Algebraic Combinatorics on Words*, page 164–196, 2002.